Limits and Continuity

Defining a Limit (Intuitive)

$\lim_{x \to a} f(x) = L$ means as x approaches a (with $x \neq a$), $f(x)$ approaches L.

Limits with Constants

If c is a constant, then

$$\lim_{x \to a} c = c \qquad \lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$$

Limit Laws

Suppose that the relevant limits exist.

$$\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$$

$$\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

$$\lim_{x \to a} \left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}, \quad \lim_{x \to a} g(x) \neq 0$$

Special Limits

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \frac{1 - \cos x}{x} = 0$$

Limits by Change of Variable

If $\lim_{t \to L} f(t)$ exists and $\lim_{x \to a} g(x) = L$ with $g(x) \neq L$ for x near a (except possibly at a itself), then

$$\lim_{x \to a} f(g(x)) = \lim_{t \to L} f(t)$$

Squeeze Theorem

If $g(x) \leqslant f(x) \leqslant h(x)$ for x near a (with $x \neq a$) and

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L\,,$$

then $\lim_{x \to a} f(x) = L$.

Definition of Continuity

- f is continuous at a point a if and only if $\lim_{x \to a} f(x) = f(a)$.
- f is continuous on (a, b) if and only if $\lim_{x \to c} f(x) = f(c)$ for all c in (a, b).
- f is continuous on $[a, b]$ if and only if f is continuous on (a, b), $\lim_{x \to a^+} f(x) = f(a)$, and $\lim_{x \to b^-} f(x) = f(b)$.

Intervals of Continuity

All elementary functions are continuous over their domains.

Function	Interval of Continuity
Polynomials: $c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$	$(-\infty, \infty)$ where n is a positive integer
Rational Expressions: $\frac{P(x)}{Q(x)}$, where P and Q are polynomials	All x such that $Q(x) \neq 0$
$\sin x$	$(-\infty, \infty)$
$\cos x$	$(-\infty, \infty)$
$\tan x$	$x \neq \frac{\pi}{2} + n\pi$ for any integer n
$\csc x$	$x \neq n\pi$
$\sec x$	$x \neq \frac{\pi}{2} + n\pi$ for any integer n
$\cot x$	$x \neq n\pi$ for any integer n
b^x, $b > 0$	$(-\infty, \infty)$
$\log_b x$, $b > 0$ and $b \neq 1$	$(0, \infty)$
$\sqrt[n]{x}$, n odd	$(-\infty, \infty)$
$\sqrt[n]{x}$, n even	$[0, \infty)$
$\lvert x \rvert$	$(-\infty, \infty)$
$\sin^{-1} x$	$[-1, 1]$
$\cos^{-1} x$	$[-1, 1]$
$\tan^{-1} x$	$(-\infty, \infty)$

Limits and Continuity

Direct Substitution

If f is continuous at a, then $\lim_{x\to a} f(x) = f(a)$.

Intermediate Value Theorem

If f is continuous on $[a, b]$, then f takes on every value between $f(a)$ and $f(b)$.

Forms of Discontinuity

- f has a removable discontinuity at a if $\lim_{x\to a} f(x)$ exists but does not equal $f(a)$, or $f(a)$ is undefined.
- f has a jump discontinuity at a if $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ equal different finite numbers.
- f has an infinite discontinuity (vertical asymptote) at a if $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ is infinite.

Continuity of Combinations of Functions

If f and g are both continuous at a, then the following functions are also continuous at a:

$$f(x) + g(x) \qquad f(x)\,g(x)$$

$$f(x) - g(x) \qquad \frac{f(x)}{g(x)},\ g(a) \neq 0$$

Continuity of Composite Functions

If $g(x)$ is continuous at $x = a$ and $f(x)$ is continuous at $x = g(a)$, then $f(g(x))$ is continuous at $x = a$—that is,

$$\lim_{x\to a} f(g(x)) = f(g(a))$$

Special Limits with Infinity

If $k > 0$ and n is a positive integer, then

$$\lim_{x\to\infty} x^k = \infty \qquad \lim_{x\to\infty} \frac{1}{x^k} = 0 \qquad \lim_{x\to-\infty} \frac{1}{x^n} = 0$$

Horizontal Asymptotes

A line $y = L$ is a horizontal asymptote to the graph of $y = f(x)$ if

$$\lim_{x\to\infty} f(x) = L \qquad \text{or} \qquad \lim_{x\to-\infty} f(x) = L$$

A graph may have zero, one, or two horizontal asymptotes.

Vertical Asymptotes

A line $x = p$ is a vertical asymptote to the graph of $y = f(x)$ if any of the following is true:

$$\lim_{x\to p^-} f(x) = \infty \qquad \lim_{x\to p^+} f(x) = \infty$$

$$\lim_{x\to p^-} f(x) = -\infty \qquad \lim_{x\to p^+} f(x) = -\infty$$

Suppose $x = p$ is a vertical asymptote to the graph of $y = f(x)$.

- If $f(x) < 0$ as $x \to p^-$, then $\lim_{x\to p^-} f(x) = -\infty$.
- If $f(x) > 0$ as $x \to p^-$, then $\lim_{x\to p^-} f(x) = \infty$.
- If $f(x) < 0$ as $x \to p^+$, then $\lim_{x\to p^+} f(x) = -\infty$.
- If $f(x) > 0$ as $x \to p^+$, then $\lim_{x\to p^+} f(x) = \infty$.

Formal Definition of a Limit

- $\lim_{x\to a} f(x) = L$ means for every $\varepsilon > 0$, a number $\delta > 0$ exists such that $|f(x) - L| < \varepsilon$ if $0 < |x - a| < \delta$.
- $\lim_{x\to a^-} f(x) = L$ means for every $\varepsilon > 0$, a number $\delta > 0$ exists such that $|f(x) - L| < \varepsilon$ if $a - \delta < x < a$.
- $\lim_{x\to a^+} f(x) = L$ means for every $\varepsilon > 0$, a number $\delta > 0$ exists such that $|f(x) - L| < \varepsilon$ if $a < x < a + \delta$.

Formal Definition of Limits with Infinity

- $\lim_{x\to\infty} f(x) = L$ means for every $\varepsilon > 0$, a number $N > 0$ exists such that $|f(x) - L| < \varepsilon$ if $x > N$.
- $\lim_{x\to-\infty} f(x) = L$ means for every $\varepsilon > 0$, a number $N < 0$ exists such that $|f(x) - L| < \varepsilon$ if $x < N$.
- $\lim_{x\to a} f(x) = \infty$ means for every $M > 0$, a number $\delta > 0$ exists such that $f(x) > M$ if $0 < |x - a| < \delta$.
- $\lim_{x\to a} f(x) = -\infty$ means for every $M < 0$, a number $\delta > 0$ exists such that $f(x) < M$ if $0 < |x - a| < \delta$.